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ENVELOPING ALGEBRAS AND COHOMOLOGY OF LEIBNIZ PAIRS

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We introduce the enveloping algebra for a Leibniz pair, and show that the category of modules over a Leibniz pair is isomorphic to the category of left modules over its enveloping algebra. Consequently, we show that the cohomology theory for a Leibniz pair introduced by Flato, Gerstenhaber, and Voronov can be interpreted by Ext-groups of modules over the enveloping algebra.

Key Words: Enveloping algebra; Leibniz pair; Leibniz pair cohomology.

2010 Mathematics Subject Classification: 16W25; 16E40.

1. INTRODUCTION

Leibniz pairs were introduced by Flato, Gerstenhaber, and Voronov in the study of deformation theory for Poisson algebras in [3]. A Leibniz pair (A, L) consists of an associative algebra A and a Lie algebra L with an action of L on A. Roughly speaking, a Leibniz pair can be viewed as an infinitesimal version of an algebra with a group of operators acting on it.

An important example of a Leibniz pair comes from a smooth manifold, especially from a Poisson or symplectic manifold, where the Lie algebra of smooth vector fields acts on the algebra of smooth functions on it. Leibniz pair also arises whenever a Lie group acts on an associative algebra. For instance, an action of a Lie group G on a smooth manifold M naturally induces an action of the Lie algebra of G on the algebra of smooth functions on M.

A cohomology theory for Leibniz pairs (LP-cohomology for short) was introduced in [3], and they showed that the LP-cohomology controls the formal deformation of Leibniz pairs. They also defined modules over a Leibniz pair.

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A natural question asked in [3] is whether the LP-cohomology can be explained by Ext-groups of modules.

In this article, we construct for each Leibniz pair (A, L) an associative algebra : $\mathcal{U}(A, L)$, called its *enveloping algebra*. We prove the following result as given in Theorem 3.5.

Theorem 1. *Let (A, L) be a Leibniz pair and $\mathcal{U}(A, L)$ be its enveloping algebra. Then the category of modules over (A, L) is isomorphic to the category of $\mathcal{U}(A, L)$-modules.*

Consequently, the category of modules over a Leibniz pair has enough projective and injective objects, which enables the usual construction of cohomology theory for a Leibniz pair by using projective or injective resolutions.

Denote by $\mathrm{H}^n_{LP}(A, L; M, P)$ the n-th LP-cohomology group of the Leibniz pair (A, L) with coefficients in an (A, L)-module (M, P, σ). By Theorem 1, the (A, L)-module (M, P, σ) corresponds to a module $(P, M, \widetilde{\sigma})$ over $\mathcal{U}(A, L)$. We consider the Ext-groups of the trivial module $(k, 0, 0)$ over $\mathcal{U}(A, L)$ in a standard way, and prove the following result, which shows that the LP-chomology is exactly interpreted by certain Ext-groups. This gives an affirmative answer to the question raised above. For more details we refer to Theorem 4.4.

Theorem 2. *Keep the above notation. Then we have isomorphisms*

$$\mathrm{H}^n_{LP}(A, L; M, P) \cong \mathrm{Ext}^n_{\mathcal{U}(A,L)}((k, 0, 0), (P, M, \widetilde{\sigma})),$$

for all $n \geq 0$.

The article is organized as follows. In Section 2, we briefly recall some basic facts on Leibniz pairs and their modules. Section 3 deals with the construction of the enveloping algebra for a Leibniz pair and a proof of Theorem 1 is given there. In Section 4, we will calculate the Ext-groups of the trivial module over a Leibniz pair and show the isomorphisms in Theorem 2. In Section 5, we will construct a long exact sequence and apply it to calculate LP cohomology groups.

2. PRELIMINARIES

Throughout k will be a fixed field of characteristic 0, all algebras considered are over k and an associative algebra A has a multiplicative identity 1_A. We write $\otimes = \otimes_k$ for simplicity.

Definition 2.1 ([3]). A *Leibniz pair* (A, L) consists of an associative algebra A and a Lie algebra L, connected by a Lie algebra homomorphism μ: $L \to \mathrm{Der}(A)$, the Lie algebra of derivations of A into itself.

Usually, elements in A will be denoted by $a, b, c, \cdots$ and those of L by $x, y, z, \cdots$. The Lie algebra homomorphism μ: $L \to \mathrm{Der}(A)$ just says that A is a Lie module over L with the action $\{-, -\}$: $L \times A \to A$ given by $\{x, a\} = \mu(x)(a)$, which satisfies the Leibniz rule

$$\{x, ab\} = a\{x, b\} + \{x, a\}b \tag{2.1}$$

for all $x \in L$ and $a, b \in A$.

Remark 2.2. Recall that a *noncommutative Poisson algebra A* is both an associative algebra and a Lie algebra with the Lie bracket $\{-,-\}$ satisfying the Leibniz rule

$$\{ab, c\} = a\{b, c\} + \{a, c\}b$$

for all $a, b, c \in A$, see also [10]. Clearly, a noncommutative Poisson algebra A corresponds to a Leibniz pair (A, A) together with the structure morphism μ given by setting $\mu(a) = \{a, -\}$ for all $a \in A$.

Definition 2.3 ([3]). Let (A, L) be a Leibniz pair. A *module* over (A, L) means a triple (M, P, σ), where P is a Lie module over L with the action $[-,-]_*: L \times P \to P$, M is both an A-A-bimodule and a Lie module over L with Lie action $\{-,-\}_*: L \times M \to M$, which satisfies

$$\{x, am\}_* = \{x, a\}m + a\{x, m\}_*, \tag{2.2}$$

$$\{x, ma\}_* = m\{x, a\} + \{x, m\}_* a, \tag{2.3}$$

for $x \in L, m \in M, a \in A$, and $\sigma: A \otimes P \to M$ is a k-linear function satisfying

$$\sigma(ab \otimes \alpha) = a\sigma(b \otimes \alpha) + \sigma(a \otimes \alpha)b \tag{2.4}$$

$$\{x, \sigma(a \otimes \alpha)\}_* = \sigma(\{x, a\} \otimes \alpha) + \sigma(a \otimes [x, \alpha]_*) \tag{2.5}$$

for $a, b \in A, \alpha \in P$ and $x \in L$.

Remark 2.4. The above definition coincides with the original one in [3]. More precisely, let P be a Lie module over L and M be an A-A-bimodule. Denote by $L \ltimes P$ (*resp.* $A \ltimes M$) the Lie (*resp.* associative) semidirect product of L and P (*resp.* A and M).

Recall that a module over (A, L) introduced in [3] means a pair (M, P), provided that P is a Lie module over L, M is an A-A-bimodule, and there is a Lie algebra homomorphism $\mathrm{H}at\mu : L \ltimes P \to \mathrm{Der}(A \ltimes M)$, which satisfies the following conditions:

(1) $\mathrm{H}at\mu((x, 0)(a, 0)) = \mu(x)(a)$ for any $x \in L, a \in A$;
(2) $\mathrm{H}at\mu((x, 0)(0, m)), \mathrm{H}at\mu((0, \alpha))((a, 0)) \in M$ for any $x \in L, a \in A, m \in M, \alpha \in P$;
(3) $\mathrm{H}at\mu((0, \alpha)(0, m)) = 0$ for any $\alpha \in P, m \in M$.

A triple (M, P, σ) corresponds to a pair (M, P) together with a Lie algebra homomorphism $\mathrm{H}at\mu : L \ltimes P \to \mathrm{Der}(A \ltimes M)$ given by

$$\mathrm{H}at\mu((x, \alpha)(a, m)) = \mu(x)(a) + \{x, m\}_* + \sigma(a \otimes \alpha)$$

for all $x \in L, \alpha \in P, a \in A, m \in M$.

A *homomorphism* (g, f): $(M, P, \sigma) \to (M', P', \sigma')$ of (A, L)-modules means that g: $M \to M'$ is a homomorphism of both A-A-bimodules and Lie modules, f: $P \to P'$ is a homomorphism of Lie modules, and the diagram

$$\begin{array}{ccc} A\otimes P & \xrightarrow{\sigma} & M \\ {\scriptstyle \mathrm{id}_A\otimes f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ A\otimes P' & \xrightarrow{\sigma'} & M' \end{array} \tag{2.6}$$

commutes. We denote the category of (A, L)-modules by $\mathcal{M}(A, L)$.

Remark 2.5. Let $(A, \cdot, \{-, -\})$ be a noncommutative Poisson algebra. Recall from [10] a *quasi-Poisson module* M over A is both an A-A-bimodule and a Lie module over A with the action given by $\{-, -\}_*: A \times M \to M$, which satisfies

$$\{a, bm\}_* = b\{a, m\}_* + \{a, b\}m,$$
$$\{a, mb\}_* = \{a, m\}_*b + m\{a, b\}$$

for all $a, b \in A$ and $m \in M$. In addition, if

$$\{ab, m\}_* = a\{b, m\}_* + \{a, m\}_*b$$

holds for all $a, b \in A$ and $m \in M$, then we say that M is a *Poisson module* over A.

Let (A, A) be the corresponding Leibniz pair. Assume that M is both an A-A-bimodule and a Lie module over A with the action given by $\{-, -\}_*: A \times M \to M$. Then

(i) M is a quasi-Poisson module over A if and only if (M, M, σ) is a module over the Leibniz pair (A, A), where σ is given by taken the commutator in the sense of associative action on M, i.e. $\sigma(a \otimes m) = am - ma$ for all $a \in A, m \in M$.
(ii) M is a Poisson module over A if and only if (M, M, σ) is a module over the Leibniz pair (A, A), where σ is given by the Lie action of A on M, i.e., $\sigma(a \otimes m) = \{a, m\}_*$ for all $a \in A, m \in M$.

Therefore, the quasi-Poisson module category and Poisson module category over A can be viewed as subcategories (but not full subcategories) of the module category over the corresponding Leibniz pair (A, A).

Denote by A^{op} the *opposite algebra* of the associative algebra A. Usually, we use a to denote an element in A and a' its counterpart in A^{op} to show the difference. Denote the enveloping algebra of A by $A^e = A \otimes A^{\mathrm{op}}$ and the universal enveloping algebra of L by $\mathcal{U}(L)$. In this article, elements in $\mathcal{U}(L)$ is written as $X, Y, Z, \cdots$ and the identity element in $\mathcal{U}(L)$ is written as $\mathbf{1}$. Note that $\mathcal{U}(L)$ is a cocommutative Hopf algebra, with the comultiplication denoted by $\Delta(X) = \sum X_{(1)} \otimes X_{(2)}$ for any $X \in \mathcal{U}(L)$, where $\sum$ is the Sweedler's notation, see [9, Section 4.0] for more details.

Suppose that P is a Lie module over L. Equivalently, P is a $\mathcal{U}(L)$-module. We denote the action $\mathcal{U}(L) \times P \to P$ as $(X, \alpha) \mapsto X(\alpha)$ for any $X \in \mathcal{U}(L)$ and $\alpha \in P$. Note that $\mathcal{U}(L)$ is a cocommutative Hopf algebra and A^e is also a $\mathcal{U}(L)$-module with the action given by

$$X(a \otimes b') = \sum X_{(1)}(a) \otimes (X_{(2)}(b))'$$

for $X \in \mathcal{U}(L)$, $a \otimes b' \in A^e$. Moreover, A^e is a $\mathcal{U}(L)$-module algebra, which means that the multiplication $A^e \otimes A^e \to A^e$ is a $\mathcal{U}(L)$-homomorphism. The smash product $A^e \sharp \mathcal{U}(L)$ is an associative algebra, see [9, Section 7.2]. Recall that $A^e \sharp \mathcal{U}(L) = A^e \otimes \mathcal{U}(L)$ as a k-vector space. The multiplication is given by

$$(a \otimes b' \sharp X)(c \otimes d' \sharp Y) = \sum aX_{(1)}(c) \otimes (X_{(2)}(d)b)' \sharp X_{(3)} Y.$$

The following lemma is straightforward, and we omit the proof here.

Lemma 2.6. *Let M be simultaneously an A-A-bimodule and a Lie module over L with the action $\{-,-\}_*: L \times M \to M$. Then M is a left $A^e \sharp \mathcal{U}(L)$-module if and only if* (2.2) *and* (2.3) *holds.*

3. ENVELOPING ALGEBRAS OF LEIBNIZ PAIRS

Let (A, L) be a Leibniz pair. We write $A^i = A^{\otimes i}$ and denote by $\Omega^1(A)$ the space of 1-forms of A, which is by definition the first syzygy of A as an A^e-module, see [8, Section 7.1]. To be precise, as an A^e-module, $\Omega^1(A) = A^3/I$, where I is a submodule of A^3 generated by

$$\{a \otimes b \otimes 1_A - 1_A \otimes ab \otimes 1_A + 1_A \otimes a \otimes b \mid a, b \in A\}.$$

We simply write the element $a_1 \otimes a_2 \otimes a_3 + I$ in $\Omega^1(A)$ as $a_1 \otimes a_2 \otimes a_3$ when no confusion can arise.

Lemma 3.1. *Let (A, L) be a Leibniz pair. The space $\Omega^1(A)$ of 1-forms is a left $A^e \sharp \mathcal{U}(L)$-module with the action given by*

$$(a \otimes b' \sharp X)(a_1 \otimes a_2 \otimes a_3) = \sum aX_{(1)}(a_1) \otimes X_{(2)}(a_2) \otimes X_{(3)}(a_3)b$$

for all $a_1 \otimes a_2 \otimes a_3 \in \Omega^1(A)$ and $a \otimes b' \sharp X \in A^e \sharp \mathcal{U}(L)$.

Proof. We consider the action of L on $\Omega^1(A)$, $\{-,-\}_*: L \times \Omega^1(A) \to \Omega^1(A)$ defined as

$$\{x, a_1 \otimes a_2 \otimes a_3\}_* = \{x, a_1\} \otimes a_2 \otimes a_3 + a_1 \otimes \{x, a_2\} \otimes a_3 + a_1 \otimes a_2 \otimes \{x, a_3\}$$

for all $x \in L$ and $a_1 \otimes a_2 \otimes a_3 \in \Omega^1(A)$. By some direct calculation, we have

$$\{x, 1_A \otimes ab \otimes 1_A\}_* = \{x, a \otimes b \otimes 1_A\}_* + \{x, 1_A \otimes a \otimes b\}_*, \tag{3.1}$$

$$\{[x, y], a_1 \otimes a_2 \otimes a_3\}_* = \{x, \{y, a_1 \otimes a_2 \otimes a_3\}_*\}_*; -\{y, \{x, a_1 \otimes a_2 \otimes a_3\}_*\}_*, \tag{3.2}$$

$$\{x, a(a_1 \otimes a_2 \otimes a_3)\}_* = a\{x, a_1 \otimes a_2 \otimes a_3\}_* + \{x, a\}(a_1 \otimes a_2 \otimes a_3), \tag{3.3}$$

$$\{x, (a_1 \otimes a_2 \otimes a_3)a\}_* = \{x, a_1 \otimes a_2 \otimes a_3\}_* a + (a_1 \otimes a_2 \otimes a_3)\{x, a\}. \tag{3.4}$$

Equality (3.1) is just to say that the action is well defined, and we know that the action gives a Lie module structure on $\Omega^1(A)$ by (3.2). It follows from Lemma 2.6 that $\Omega^1(A)$ is an $A^e \sharp \mathcal{U}(L)$-module by (3.3) and (3.4). □

We denote $\overline{\Omega} = \Omega^1(A) \otimes \mathcal{U}(L)$, which is an $(A^e \sharp \mathcal{U}(L))$-$\mathcal{U}(L)$-bimodule.

Lemma 3.2. *Keep the above notation, and let σ: $A \otimes P \to M$ be a k-linear map. Then the map*

$$\widetilde{\sigma}: \overline{\Omega} \underset{\mathcal{U}(L)}{\otimes} P \to M, \qquad \widetilde{\sigma}((a_1 \otimes a_2 \otimes a_3 \otimes X) \otimes \alpha) = a_1 \sigma(a_2 \otimes X(\alpha)) a_3$$

is an $A^e \sharp \mathcal{U}(L)$-homomorphism if and only if σ satisfies (2.4) *and* (2.5).

Proof. Assume that σ satisfies (2.4) and (2.5). By definition, we know that

$$\widetilde{\sigma}((a_1 \otimes a_2 \otimes a_3 \otimes X) \otimes \alpha) = \widetilde{\sigma}((a_1 \otimes a_2 \otimes a_3 \otimes \mathbf{1}) \otimes X(\alpha)),$$

and by (2.4),

$$\begin{aligned}
&\widetilde{\sigma}((1_A \otimes ab \otimes 1_A \otimes X) \otimes \alpha) \\
&\quad = \sigma(ab \otimes X(\alpha)) \\
&\quad = a\sigma(b \otimes X(\alpha)) + \sigma(a \otimes X(\alpha))b \\
&\quad = \widetilde{\sigma}((a \otimes b \otimes 1_A \otimes X) \otimes \alpha) + \widetilde{\sigma}((1_A \otimes a \otimes b \otimes X) \otimes \alpha).
\end{aligned}$$

It follows that $\widetilde{\sigma}$ is well defined.

By direct calculation, we have

$$\begin{aligned}
&\widetilde{\sigma}((a \otimes b' \sharp X)(a_1 \otimes a_2 \otimes a_3 \otimes Y \otimes \alpha) \\
&\quad = \sum \widetilde{\sigma}(aX_{(1)}(a_1) \otimes X_{(2)}(a_2) \otimes X_{(3)}(a_3)b \otimes X_{(4)}Y \otimes \alpha) \\
&\quad = \sum aX_{(1)}(a_1)\widetilde{\sigma}ma(X_{(2)}(a_2) \otimes X_{(4)}Y(\alpha))X_{(3)}(a_3)b.
\end{aligned} \tag{3.5}$$

On the other hand,

$$\begin{aligned}
&(a \otimes b' \sharp X)\widetilde{\sigma}(a_1 \otimes a_2 \otimes a_3 \otimes Y \otimes \alpha) \\
&\quad = (a \otimes b' \sharp X)(a_1 \widetilde{\sigma}ma(a_2 \otimes Y(\alpha))a_3) \\
&\quad = ((a \otimes b' \sharp X)(a_1 \otimes a_3' \sharp \mathbf{1}))\widetilde{\sigma}ma(a_2 \otimes Y(\alpha)) \\
&\quad = \sum (aX_{(1)}(a_1) \otimes X_{(2)}(a_3)b \sharp X_{(3)})\widetilde{\sigma}ma(a_2 \otimes Y(\alpha)) \\
&\quad = \sum aX_{(1)}(a_1)X_{(3)}(\widetilde{\sigma}ma(a_2 \otimes Y(\alpha)))X_{(2)}(a_3)b \\
&\quad = \sum aX_{(1)}(a_1)\widetilde{\sigma}ma(X_{(3)_{(1)}}(a_2) \otimes X_{(3)_{(2)}}(Y(\alpha)))X_{(2)}(a_3)b \\
&\quad = (3.5),
\end{aligned}$$

where the last equality is deduced from the cocommutativity of $\mathcal{U}(L)$. Consequently, $\widetilde{\sigma}$ is a homomorphism of $A^e \sharp \mathcal{U}(L)$-modules.

Conversely, if $\widetilde{\sigma}$ is an $A^e \sharp \mathcal{U}(L)$-homomorphism, it is easily checked that $\widetilde{\sigma}ma$ satisfies (2.4) and (2.5). $\square$

Definition 3.3. Let (A, L) be a Leibniz pair. The triangular matrix algebra

$$\begin{pmatrix} \mathcal{U}(L) & 0 \\ \overline{\Omega} & A^e\sharp\mathcal{U}(L) \end{pmatrix}$$

is called the *enveloping algebra* of (A, L), denoted by $\mathcal{U}(A, L)$.

Remark 3.4. A module $(P, M, \widetilde{\sigma})$ over $\mathcal{U}(A, L)$ means that P is a $\mathcal{U}(L)$-module, M is an $A^e\sharp\mathcal{U}(L)$-module, and $\widetilde{\sigma} : \overline{\Omega} \underset{\mathcal{U}(L)}{\otimes} P \to M$ is a homomorphism of $A^e\sharp\mathcal{U}(L)$-modules. A homomorphism (f, g): $(P, M, \widetilde{\sigma}) \to (P', M', \widetilde{\sigma}')$ of $\mathcal{U}(A, L)$-modules means that $f : P \to P'$ is a $\mathcal{U}(L)$-homomorphism, $g : M \to M'$ is an $A^e\sharp\mathcal{U}(L)$-homomorphism, and the following diagram commutes:

$$\begin{array}{ccc} \overline{\Omega} \underset{\mathcal{U}(L)}{\otimes} P & \xrightarrow{\widetilde{\sigma}} & M \\ {\scriptstyle \mathrm{id}_{\overline{\Omega}}\otimes f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \overline{\Omega} \underset{\mathcal{U}(L)}{\otimes} P' & \xrightarrow[\widetilde{\sigma}']{} & M' \end{array} \tag{3.6}$$

Denote by $\mathcal{U}(A, L)$-Mod the category of $\mathcal{U}(A, L)$-modules.

Theorem 3.5. *Let (A, L) be a Leibniz pair. Then the category of modules over (A, L) is isomorphic to the category of $\mathcal{U}(A, L)$-modules.*

Proof. First, we define a functor F: $\mathcal{M}(A, L) \to \mathcal{U}(A, L)$-Mod as follows. Suppose that $(M, P, \widetilde{\sigma}ma)$ is a module over the Leibniz pair (A, L). We define $F((M, P, \widetilde{\sigma}ma)) = (P, M, \widetilde{\sigma})$ with the action of $\mathcal{U}(A, L)$ given by setting

$$\begin{pmatrix} X & 0 \\ a_1 \otimes a_2 \otimes a_3 \otimes Z & a \otimes b'\sharp Y \end{pmatrix} \begin{pmatrix} \alpha \\ m \end{pmatrix} = \begin{pmatrix} X(\alpha) \\ \widetilde{\sigma}(a_1 \otimes a_2 \otimes a_3 \otimes Z \otimes \alpha) + a(Y(m))b \end{pmatrix},$$

where $\widetilde{\sigma}$ is given by Lemma 3.2, i.e.,

$$\widetilde{\sigma}(a_1 \otimes a_2 \otimes a_3 \otimes Z \otimes \alpha) = a_1\widetilde{\sigma}ma(a_2 \otimes Z(\alpha))a_3$$

for all $a_1 \otimes a_2 \otimes a_3 \otimes Z \otimes \alpha \in \overline{\Omega} \underset{\mathcal{U}(L)}{\otimes} P$. By Lemma 3.2, we have $\widetilde{\sigma}$: $\overline{\Omega} \underset{\mathcal{U}(L)}{\otimes} P \to M$ is a homomorphism of $A^e\sharp\mathcal{U}(L)$-modules, and hence the triple $(P, M, \widetilde{\sigma})$ is a module over $\mathcal{U}(A, L)$.

For a homomorphism (g, f): $(M, P, \widetilde{\sigma}ma) \to (M', P', \widetilde{\sigma}ma')$ of (A, L)-modules, we define $F((g, f)) = (f, g)$. From the commutativity of the diagram (2.6), it follows that the diagram (3.6) commutes. Therefore, (f, g): $(P, M, \widetilde{\sigma}) \to (P', M', \widetilde{\sigma}')$ is a $\mathcal{U}(A, L)$-homomorphism.

On the other hand, we define a functor G: $\mathcal{U}(A, L)$-Mod $\to \mathcal{M}(A, L)$ as follows. For each left $\mathcal{U}(A, L)$-module $(P, M, \widetilde{\sigma})$, $G((P, M, \widetilde{\sigma})) = (M, P, \widetilde{\sigma}ma)$, where P is a $\mathcal{U}(L)$-module and hence a Lie module over L, and M is an $A^e\sharp\mathcal{U}(L)$-module. By Lemma 2.6, M is simultaneously an A-A-bimodule and a Lie module over L

satisfying (2.2) and (2.3). It follows from Lemma 3.2 that the corresponding triple $(M, P, \widetilde{\sigma}ma)$ is a module over the Leibniz pair (A, L).

For any $\mathcal{U}(A, L)$-homomorphism (f, g): $(P, M, \widetilde{\sigma}) \to (P', M', \widetilde{\sigma}')$, it is easy to check that $G((f, g)) = (g, f)$ is a homomorphism of (A, L)-modules from $(M, P, \widetilde{\sigma}ma)$ to $(M', P', \widetilde{\sigma}ma')$ because the diagram (2.6) is commutative if and only if the diagram (3.6) commutes.

The functors F and G are mutually inverse. $\square$

4. COHOMOLOGY FOR LEIBNIZ PAIRS

Let (A, L) be a Leibniz pair. We write $\wedge^j L = \wedge^j$ for short.

Theorem 3.5 implies that the module category over a Leibniz pair (A, L) has enough projective and injective objects, which enables us to construct the cohomology theory for Leibniz pairs by using projective or injective resolution in a standard way.

We begin with a well-known result concerning projective modules over a general matrix triangular algebra.

Lemma 4.1 ([1, Proposition 2.5]). *Let $\Lambda = \begin{pmatrix} A & 0 \\ {}_BM_A & B \end{pmatrix}$ be a triangular matrix algebra. Then $(P, Q, \widetilde{\sigma}ma)$ is a projective Λ-module if and only if P is a projective A-module and $\widetilde{\sigma}ma$: $M \otimes_A P \to Q$ is a split monomorphism of B-modules with* Coker$(\widetilde{\sigma}ma)$ *being a projective B-module.*

We come back to the Leibniz pair (A, L). Consider the projective resolution of the trivial $\mathcal{U}(L)$-module k

$$\mathbb{K}_\bullet \qquad \cdots \to \mathcal{U}(L) \otimes \wedge^j \xrightarrow{d_j} \mathcal{U}(L) \otimes \wedge^{j-1} \to \cdots \to \mathcal{U}(L) \otimes \wedge^1 \xrightarrow{d_1} \mathcal{U}(L) \to 0,$$

where

$$\begin{aligned} & d_j(X \otimes x_1 \wedge x_2 \wedge \cdots \wedge x_j) \\ & \quad = \sum_{k=1}^{j} (-1)^{k-1} X(x_k) \otimes x_1 \wedge \cdots \widehat{x_k} \cdots \wedge x_j \\ & \quad\quad + \sum_{1 \le p < q \le j} (-1)^{p+q} X \otimes [x_p, x_q] \wedge x_1 \wedge \cdots \widehat{x_p} \cdots \widehat{x_q} \cdots \wedge x_j \end{aligned}$$

for all $X \otimes x_1 \wedge x_2 \wedge \cdots \wedge x_j \in \mathcal{U}(L) \otimes \wedge^j$, $j \geq 1$, [5, Chapter VII, Theorem 4.2]. The standard resolution of $\Omega^1(A)$ as an A^e-module is given as

$$\mathbb{S}_\bullet \qquad \cdots \to A^{i+3} \xrightarrow{\delta_i} A^{i+2} \to \cdots \to A^4 \xrightarrow{\delta_1} A^3 \xrightarrow{\delta_0} \Omega^1(A) \to 0,$$

where

$$\delta_i(a_1 \otimes a_2 \otimes \cdots \otimes a_{i+3}) = \sum_{k=1}^{i+2} (-1)^{i-1} a_1 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_{i+3}$$

for all $a_1 \otimes a_2 \otimes \cdots \otimes a_{i+3} \in A^{i+3}$, $i \geq 1$ and δ_0 is the canonical projection, [4].

Taking tensor product $\mathbb{K}_\bullet \otimes \mathbb{S}_\bullet$, we obtain a bicomplex

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & A^4 \otimes \mathcal{U}(L) & \xleftarrow{\delta^H_{2,0}} & A^4 \otimes \mathcal{U}(L) \otimes \wedge^1 & \xleftarrow{\delta^H_{2,1}} & A^4 \otimes \mathcal{U}(L) \otimes \wedge^2 & \longleftarrow & \cdots \\
 & & \delta^V_{1,0}\downarrow & & \delta^V_{1,1}\downarrow & & \delta^V_{1,2}\downarrow & & \\
0 & \longleftarrow & A^3 \otimes \mathcal{U}(L) & \xleftarrow{\delta^H_{1,0}} & A^3 \otimes \mathcal{U}(L) \otimes \wedge^1 & \xleftarrow{\delta^H_{1,1}} & A^3 \otimes \mathcal{U}(L) \otimes \wedge^2 & \longleftarrow & \cdots \\
 & & \delta^V_{0,0}\downarrow & & \delta^V_{0,1}\downarrow & & \delta^V_{0,2}\downarrow & & \\
0 & \longleftarrow & \Omega^1(A) \otimes \mathcal{U}(L) & \xleftarrow{\delta^H_{0,0}} & \Omega^1(A) \otimes \mathcal{U}(L) \otimes \wedge^1 & \xleftarrow{\delta^H_{0,1}} & \Omega^1(A) \otimes \mathcal{U}(L) \otimes \wedge^2 & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\delta^H_{i,j} = \mathrm{id} \otimes d_j$, and $\delta^V_{i,j} = \delta_i \otimes \mathrm{id}$. This is a bicomplex of $A^e \sharp \mathcal{U}(L)$-modules.

We denote $T_i = A^{i+3} \otimes \mathcal{U}(L)$ for $i \geq 0$, $T_{-1} = \overline{\Omega}$, and $K_j = \mathcal{U}(L) \otimes \wedge^j$ for $j \geq 0$. By Künneth's Theorem [5, Chapter V, Theorem 2.1], the total complex of the bicomplex, denoted by $\mathbb{Q}_\bullet$,

$$\cdots \to \bigoplus_{i=0}^{n} T_{i-1} \otimes \wedge^{n-i} \xrightarrow{\varphi_n} \bigoplus_{i=0}^{n-1} T_{i-1} \otimes \wedge^{n-i-1} \to \cdots \to T_0 \oplus T_{-1} \otimes \wedge^1 \xrightarrow{\varphi_0} T_{-1} \to 0$$

is exact, where $\varphi_n = \sum_{i+j=n} \delta^H_{i,j} + (-1)^i \delta^V_{i,j}$ for $n \geq 0$.

Lemma 4.2. *Using the above notation, we have that*

$$\mathbb{P}_\bullet \quad \cdots \to \left(K_n, \bigoplus_{i=0}^{n} T_{i-1} \otimes \wedge^{n-i}, \iota_n\right) \xrightarrow{(d_n, \varphi_n)} \left(K_{n-1}, \bigoplus_{i=0}^{n-1} T_{i-1} \otimes \wedge^{n-i-1}, \iota_{n-1}\right)$$
$$\to \cdots \to (K_0, T_{-1}, \iota_0) \to 0$$

is a projective resolution of $(k, 0, 0)$ *as a* $\mathcal{U}(A, L)$*-module, where*

$$\iota_n : \overline{\Omega} \underset{\mathcal{U}(L)}{\otimes} K_n \to \bigoplus_{i=0}^{n} T_{i-1} \otimes \wedge^{n-i}$$
$$(a_1 \otimes a_2 \otimes a_3 \otimes X) \otimes (Y \otimes \omega) \mapsto (a_1 \otimes a_2 \otimes a_3 \otimes XY) \otimes \omega$$

for $n \geq 0$.

Proof. Note that $T_i \underset{\mathcal{U}(L)}{\otimes} K_j$ is isomorphic to $A^{i+3} \otimes \mathcal{U}(L) \otimes \wedge^j$, which is a free $A^e \sharp \mathcal{U}(L)$-module for $i, j \geq 0$. The $\mathcal{U}(L)$-module K_n is free and ι_n is a split monomorphism with $\mathrm{Coker}(\iota_n)$ being projective, since ι_n is the composition of the natural isomorphism $\overline{\Omega} \otimes LK_n \cong \overline{\Omega} \otimes \wedge^n$ and the inclusion map $\overline{\Omega} \otimes \wedge^n \hookrightarrow \bigoplus_{i=0}^{n} T_{i-1} \otimes \wedge^{n-i}$. It follows from Lemma 4.1 that $(K_n, \bigoplus_{i=0}^{n} T_{i-1} \otimes \wedge^{n-i}, \iota_n)$ is a projective $\mathcal{U}(A, L)$-module.

By direct calculation, we have that the diagram

$$\begin{array}{ccc}
\overline{\Omega}\otimes K_n & \xrightarrow{\mathrm{id}\otimes d_n} & \overline{\Omega}\otimes K_{n-1} \\
\downarrow{\scriptstyle \iota_n} & & \downarrow{\scriptstyle \iota_{n-1}} \\
\bigoplus_{i=0}^{n} T_{i-1}\otimes\wedge^{n-i} & \xrightarrow[\varphi_n]{} & \bigoplus_{i=0}^{n-1} T_{i-1}\otimes\wedge^{n-i}
\end{array}$$

is commutative and (d_n, φ_n) is a homomorphism of $\mathscr{U}(A, L)$-modules. By the exactness of $\mathbb{K}_\bullet \to k \to 0$ and the complex $\mathbb{Q}_\bullet$, we know that $\mathbb{P}_\bullet$ is a projective resolution of the trivial $\mathscr{U}(A, L)$-module $(k, 0, 0)$. □

Lemma 4.3. *Let $(P, M, \widetilde{\sigma})$ be a module over $\mathscr{U}(A, L)$. Then*

$$\mathrm{H}om_{\mathscr{U}(A,L)}\left(\left(K_n, \bigoplus_{i=0}^{n} T_{i-1}\otimes\wedge^{n-i}, \iota_n\right), (P, M, \widetilde{\sigma})\right) \cong \mathrm{H}om_k(\wedge^n, P)\oplus\left(\bigoplus_{i=1}^{n}\mathrm{H}om_k(A^i\otimes\wedge^{n-i}, M)\right).$$

Proof. By definition, a pair (f, g) is a $\mathscr{U}(A, L)$-homomorphism from $(K_n, \bigoplus_{i=0}^{n} T_{i-1}\otimes \wedge^{n-i}, \iota_n)$ to $(P, M, \widetilde{\sigma})$ if and only if $f \in \mathrm{H}om_{\mathscr{U}(L)}(K_n, P)$, $g \in \bigoplus_{i=0}^{n}\mathrm{H}om_{A^e\sharp\mathscr{U}(L)}(T_{i-1}\otimes \wedge^{n-i}, M)$, and the diagram

$$\begin{array}{ccc}
\overline{\Omega}\underset{\mathcal{U}(L)}{\otimes} K_n & \xrightarrow{\mathrm{id}_{\overline{\Omega}}\otimes f} & \overline{\Omega}\underset{\mathcal{U}(L)}{\otimes} P \\
\downarrow{\scriptstyle \iota_n} & & \downarrow{\scriptstyle \widetilde{\sigma}} \\
\bigoplus_{i=0}^{n} T_{i-1}\otimes\wedge^{n-i} & \xrightarrow[g]{} & M
\end{array}$$

commutes. Write $g = (g_n, \cdots, g_1, g_0)$ with $g_i \in \mathrm{H}om_{A^e\sharp\mathscr{U}(L)}(T_{i-1}\otimes\wedge^{n-i}, M)$, $i \geq 0$. The commutativity of the diagram reads as $g_0 = g\circ\iota_n = \widetilde{\sigma}\circ(\mathrm{id}_{\overline{\Omega}}\otimes f)$. Thus (f, g) is uniquely determined by $(f, g_n, \cdots, g_1)$.

Moreover, we have isomorphisms of k-vector spaces

$$\mathrm{H}om_{\mathscr{U}(L)}(K_n, P) \cong \mathrm{H}om_k(\wedge^n, P)$$

and

$$\mathrm{H}om_{A^e\sharp\mathscr{U}(L)}(T_{i-1}\otimes\wedge^{n-i}, M) \cong \mathrm{H}om_k(A^i\otimes\wedge^{n-i}, M)$$

for any $n \geq 0$ and $i \geq 1$. Therefore, there is an isomorphism of the k-vector spaces

$$\mathrm{H}om_{\mathscr{U}(A,L)}((K_n, \bigoplus_{0\leq i\leq n} T_{i-1}\otimes\wedge^{n-i}, \iota_n), (P, M, \widetilde{\sigma})) \cong \mathrm{H}om_k(\wedge^n, P)\oplus\left(\bigoplus_{i=1}^{n}\mathrm{H}om_k(A^i\otimes\wedge^{n-i}, M)\right).$$ □

Recall the cohomology group $\mathrm{H}^{\bullet}_{LP}(A, L; M, P)$ of the Leibniz pair (A, L) with coefficients in the module $(M, P, \widetilde{\sigma}ma)$, which is defined as the cohomology group of the total complex of the following bicomplex $C^{\bullet,\bullet}(A, L; M, P)$, see [3] for detail:

$$\begin{tikzcd}
 & \cdots & \cdots & \cdots \\
0 \arrow[r] & \mathrm{Hom}_k(A^2, M) \arrow[u] \arrow[r, "\delta_H"] & \mathrm{Hom}_k(A^2 \otimes \wedge^1, M) \arrow[u] \arrow[r, "\delta_H"] & \mathrm{Hom}_k(A^2 \otimes \wedge^2, M) \cdots \arrow[u] \\
0 \arrow[r] & \mathrm{Hom}_k(A, M) \arrow[u, "\delta_V"] \arrow[r, "\delta_H"] & \mathrm{Hom}_k(A \otimes \wedge^1, M) \arrow[u, "\delta_V"] \arrow[r, "\delta_H"] & \mathrm{Hom}_k(A \otimes \wedge^2, M) \cdots \arrow[u, "\delta_V"'] \\
0 \arrow[r] & P \arrow[u, "\delta_v"] \arrow[r, "\delta_h"] & \mathrm{Hom}_k(\wedge^1, P) \arrow[u, "\delta_v"] \arrow[r, "\delta_h"] & \mathrm{Hom}_k(\wedge^2, P) \quad \cdots \arrow[u, "\delta_v"'] \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

where $\delta_v : \mathrm{H}om_k(\wedge^j, P) \to \mathrm{H}om_k(A \otimes \wedge^j, M)$,

$$(\delta_v f)(a \otimes \omega) = \widetilde{\sigma}ma(a \otimes f(\omega)),$$

$\delta_V : \mathrm{H}om_k(A^i \otimes \wedge^j, M) \to \mathrm{H}om_k(A^{i+1} \otimes \wedge^j, M)$,

$$\begin{aligned}
&\delta_V(f)(a_0 \otimes a_1 \otimes \cdots \otimes a_i \otimes \omega) \\
&\quad = a_0 f(a_1 \otimes \cdots \otimes a_i \otimes \omega) \\
&\qquad + \sum_{l=0}^{i-1} (-1)^{l+1} f(a_0 \otimes \cdots \otimes a_{l-1} \otimes a_l a_{l+1} \otimes a_{l+2} \otimes \cdots \otimes a_i \otimes \omega) \\
&\qquad + (-1)^{i+1} f(a_0 \otimes \cdots \otimes a_{i-1} \otimes \omega) a_i,
\end{aligned}$$

$\delta_H : \mathrm{H}om_k(A^i \otimes \wedge^j, M) \to \mathrm{H}om_k(A^i \otimes \wedge^{j+1}, M)$,

$$\begin{aligned}
&\delta_H(f)(a_1 \otimes \cdots \otimes a_i \otimes x_0 \wedge \cdots \wedge x_j) \\
&\quad = \sum_{l=0}^{j} (-1)^l \big(\{x_l, f(a_1 \otimes \cdots \otimes a_i \otimes x_0 \wedge \cdots \widehat{x_l} \cdots \wedge x_j)\}_* \\
&\qquad - \sum_{t=1}^{i} f(a_1 \otimes \cdots \otimes a_{t-1} \otimes \{x_l, a_t\} \otimes a_{t+1} \otimes \cdots \otimes a_i \otimes x_0 \wedge \cdots \widehat{x_l} \cdots \wedge x_j) \big) \\
&\qquad + \sum_{0 \le p < q \le j} (-1)^{p+q} f(a_1 \otimes \cdots \otimes a_i \otimes [x_p, x_q] \wedge x_0 \wedge \cdots \widehat{x_p} \cdots \widehat{x_q} \cdots \wedge x_j),
\end{aligned}$$

and δ_h: $\mathrm{H}om_k(\wedge^n, P) \to \mathrm{H}om_k(\wedge^{n+1}, P)$ is just the Chevalley–Eilenberg coboundary, i.e.,

$$\begin{aligned}
&(\delta_h f)(x_1 \wedge \cdots \wedge x_{n+1}) \\
&\quad = \sum_{i=1}^{n+1}(-1)^{i-1}[x_i, f(x_1 \wedge \cdots \widehat{x_i} \cdots \wedge x_{n+1})]_* \\
&\qquad + \sum_{1 \le p < q \le n+1} (-1)^{p+q} f([x_p, x_q] \wedge x_1 \wedge \cdots \widehat{x_p} \cdots \widehat{x_q} \cdots \wedge x_{n+1}).
\end{aligned}$$

When $M = A$, $P = L$, the Leibniz pair cohomology $\mathrm{H}^n_{LP}(A, L; A, L)$ is denoted by $\mathrm{H}^n_{LP}(A, L)$ for short. We introduce the following main result.

Theorem 4.4. *Let (A, L) be a Leibniz pair and $\mathscr{U}(A, L)$ be the enveloping algebra of (A, L). If $(M, P, \widetilde{\sigma}ma)$ is a module over (A, L) and $(P, M, \widetilde{\sigma})$ is its corresponding $\mathscr{U}(A, L)$-module, then*

$$\mathrm{H}^n_{LP}(A, L; M, P) \cong \mathrm{Ext}^n_{\mathscr{U}(A,L)}((k, 0, 0), (P, M, \widetilde{\sigma})).$$

Proof. Use the notation in Lemma 4.2. It follows from Lemma 4.2 that

$$\mathrm{Ext}^n_{\mathscr{U}(A,L)}((k, 0, 0), (P, M, \widetilde{\sigma})) \cong \mathrm{H}^n\mathrm{H}om(\mathbb{P}_\bullet, (P, M, \widetilde{\sigma}))$$

for any $\mathscr{U}(A, L)$-module $(P, M, \widetilde{\sigma})$. By simple calculation, we know that the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{U}(A,L)}(\mathcal{P}_{n-1}, (P, M, \widetilde{\sigma})) \arrow[r, "(d_n, \varphi_n)^*"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathcal{U}(A,L)}(\mathcal{P}_n, (P, M, \widetilde{\sigma})) \arrow[d, "\cong"] \\
\bigoplus\limits_{i+j=n-1} C^{i,j}(A, L; M, P) \arrow[r] & \bigoplus\limits_{i+j=n} C^{i,j}(A, L; M, P)
\end{tikzcd}$$

is commutative, where $\mathscr{P}_n = (K_n, \overset{n}{\mathscr{U}nderseti} = 0 \oplus T_{i-1} \otimes \wedge^{n-i}, \iota_n)$, and the vertical isomorphisms are given by the proof of Lemma 4.3. It follows that the total complex of the bicomplex $C^{\bullet,\bullet}(A, L; M, P)$ is isomorphic to the complex $\mathrm{H}om(\mathbb{P}_\bullet, (P, M, \widetilde{\sigma}))$, and hence

$$\mathrm{H}^n_{LP}(A, L; M, P) \cong \mathrm{Ext}^n_{\mathscr{U}(A,L)}((k, 0, 0), (P, M, \widetilde{\sigma})). \qquad \square$$

5. A LONG EXACT SEQUENCE

In this section, we give a long exact sequence and apply it to characterize the Leibniz pair cohomology.

Consider the bicomplex

$$\begin{tikzcd}
 & \cdots & \cdots & \cdots \\
0 \arrow[r] & \mathrm{Hom}_k(A^3, M) \arrow[r, "\delta_H"] \arrow[u] & \mathrm{Hom}_k(A^3 \otimes \wedge^1, M) \arrow[r, "\delta_H"] \arrow[u] & \mathrm{Hom}_k(A^3 \otimes \wedge^2, M) \cdots \arrow[u] \\
0 \arrow[r] & \mathrm{Hom}_k(A^2, M) \arrow[r, "\delta_H"] \arrow[u, "\delta_V"] & \mathrm{Hom}_k(A^2 \otimes \wedge^1, M) \arrow[r, "\delta_H"] \arrow[u, "\delta_V"] & \mathrm{Hom}_k(A^2 \otimes \wedge^2, M) \cdots \arrow[u, "\delta_V"'] \\
0 \arrow[r] & \mathrm{Hom}_k(A, M) \arrow[r, "\delta_H"] \arrow[u, "\delta_V"] & \mathrm{Hom}_k(A \otimes \wedge^1, M) \arrow[r, "\delta_H"] \arrow[u, "\delta_V"] & \mathrm{Hom}_k(A \otimes \wedge^2, M) \cdots \arrow[u, "\delta_V"'] \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

which is a sub-bicomplex of $C^{\bullet,\bullet}(A, L; M, P)$ and denoted by $Q^{\bullet,\bullet}(A, L; M)$.

Lemma 5.1. *Keeping the above notation, we have*

$$\mathrm{H}^n\mathrm{Tot}(Q^{\bullet,\bullet}(A, L; M)) \cong \mathrm{Ext}^n_{A^e \sharp \mathcal{U}(L)}(\Omega^1(A), M).$$

Proof. Consider the standard resolution of the A^e-module $\Omega^1(A)$

$$\cdots \to A^{i+3} \xrightarrow{\delta_i} A^{i+2} \to \cdots \to A^4 \xrightarrow{\delta_1} A^3 \to 0,$$

and the projective resolution of trivial $\mathcal{U}(L)$-module k

$$\cdots \to \mathcal{U}(L) \otimes \wedge^j \xrightarrow{d_j} \mathcal{U}(L) \otimes \wedge^{j-1} \to \cdots \to \mathcal{U}(L) \xrightarrow{d_1} \mathcal{U}(L) \to 0.$$

Taking the tensor product of these resolutions, we obtain the following bicomplex, denoted by $Q_{\bullet,\bullet}(A, L)$,

$$\begin{tikzcd}
 & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \\
0 & A^5 \otimes \mathcal{U}(L) \arrow[l] \arrow[d, "\delta^V_{2,0}"'] & A^5 \otimes \mathcal{U}(L) \otimes \wedge^1 \arrow[l, "\delta^H_{3,0}"'] \arrow[d, "\delta^V_{2,1}"'] & A^5 \otimes \mathcal{U}(L) \otimes \wedge^2 \arrow[l, "\delta^H_{3,1}"'] \arrow[d, "\delta^V_{2,2}"'] & \cdots \arrow[l] \\
0 & A^4 \otimes \mathcal{U}(L) \arrow[l] \arrow[d, "\delta^V_{1,0}"'] & A^4 \otimes \mathcal{U}(L) \otimes \wedge^1 \arrow[l, "\delta^H_{2,0}"'] \arrow[d, "\delta^V_{1,1}"'] & A^4 \otimes \mathcal{U}(L) \otimes \wedge^2 \arrow[l, "\delta^H_{2,1}"'] \arrow[d, "\delta^V_{1,2}"'] & \cdots \arrow[l] \\
0 & A^3 \otimes \mathcal{U}(L) \arrow[l] \arrow[d] & A^3 \otimes \mathcal{U}(L) \otimes \wedge^1 \arrow[l, "\delta^H_{1,0}"'] \arrow[d] & A^3 \otimes \mathcal{U}(L) \otimes \wedge^2 \arrow[l, "\delta^H_{1,1}"'] \arrow[d] & \cdots \arrow[l] \\
 & 0 & 0 & 0 &
\end{tikzcd}.$$

The following argument is similar to the calculation of quasi-Poisson cohomology groups in [2, Theorem 3.7]. Since $A^{i+2} \otimes \mathcal{U}(L) \otimes \wedge^j$ is free as an $A^e \sharp \mathcal{U}(L)$-module for $i, j \geq 0$, we know $\mathrm{H}_n(\mathrm{Tot}(Q_{\bullet,\bullet}(A, L))) = 0$ for $n \geq 1$ and $\mathrm{H}_0(\mathrm{Tot}(Q_{\bullet,\bullet}(A, L))) = \Omega^1(A)$. The complex $\mathrm{Tot}(Q_{\bullet,\bullet}(A, L))$ is a projective resolution of $\Omega^1(A)$ as an $A^e \sharp \mathcal{U}(L)$-module. Applying the functor $\mathrm{H}om_{A^e \sharp \mathcal{U}(L)}(-, M)$ on $Q_{\bullet,\bullet}(A, L)$ and the k-linear isomorphism

$$\mathrm{H}om_{A^e \sharp \mathcal{U}(L)}(A^{i+2} \otimes \mathcal{U}(L) \otimes \wedge^j, M) \cong \mathrm{H}om_k(A^i \otimes \wedge^j, M),$$

we immediately get the bicomplex $Q^{\bullet,\bullet}(A, L; M)$. Consequently,

$$\mathrm{H}^n\mathrm{Tot}(Q^{\bullet,\bullet}(A, L; M)) \cong \mathrm{Tot}\mathrm{H}^n Q^{\bullet,\bullet}(A, L; M) \cong \mathrm{Ext}^n_{A^e \sharp \mathcal{U}(L)}(\Omega^1(A), M). \qquad \square$$

Remark 5.2. Applying a general result for smash products, see [2, Theorem 5.2] for details, we have a Grothendieck spectral sequence

$$\mathrm{Ext}^q_{\mathcal{U}(L)}(k, \mathrm{Ext}^p_{A^e}(\Omega^1(A), M)) \Longrightarrow \mathrm{Ext}^{p+q}_{A^e \sharp \mathcal{U}(L)}(\Omega^1(A), M). \tag{5.1}$$

For some special cases, it can be used to calculate the Ext-group at the right side.

Theorem 5.3. *Let (A, L) be a Leibniz pair and $(M, P, \widetilde{\sigma}ma)$ be a module over (A, L). Then we have the long exact sequence*

$$\begin{aligned}
0 &\to \mathrm{H}om_{A^e \sharp \mathcal{U}(L)}(\Omega^1(A), M) \to \mathrm{H}^0_{LP}(A, L; M, P) \to \mathrm{H}L^0(L, P) \\
&\to \mathrm{Ext}^1_{A^e \sharp \mathcal{U}(L)}(\Omega^1(A), M) \to \mathrm{H}^1_{LP}(A, L; M, P) \to \mathrm{H}L^1(L, P) \to \cdots \\
&\to \mathrm{H}^n_{LP}(A, L; M, P) \to \mathrm{H}L^n(L, P) \to \mathrm{Ext}^{n+1}_{A^e \sharp \mathcal{U}(L)}(\Omega^1(A), M) \to \cdots,
\end{aligned}$$

where $\mathrm{H}L^n(L, P)$ is the nth cohomology group of the Lie algebra L with coefficients in P.

Proof. By the bicomplex used to define Leibniz pair cohomology, we have a short exact sequence of complexes

$$0 \to \mathrm{Tot}(Q^{\bullet,\bullet}(A, L; M)) \to \mathrm{Tot}(C^{\bullet,\bullet}(A, L; M, P)) \to \mathrm{H}om_k(\wedge^\bullet, P) \to 0.$$

By the long exact sequence theorem and Lemma 5.1, we have the long exact sequence. $\square$

There are some simple observations about LP-cohomology group from Theorem 5.3.

Corollary 5.4. *Let (A, L) be a Leibniz pair and (M, P) be a module over (A, L). If A, L are finite-dimensional, and $\mathrm{gl.dim}A < \infty$, then $\mathrm{H}^n_{LP}(A, L; M, P) = 0$ for sufficiently large n.*

Proof. If the associative algebra A is finite-dimensional and $\mathrm{gl.dim}A < \infty$, then there exists $p > 0$ such that $\mathrm{Ext}^n_{A^e}(\Omega^1(A), M) = 0$ for all $n \geq p$ since proj.dim$_{A^e}A = $ gl.dim$A$, see [4, Section 1.5]. On the other hand, $\wedge^q = 0$ and hence $\mathrm{H}L^q(L, N) = \mathrm{Ext}^q_{\mathcal{U}(L)}(k, N) = 0$ for any $q > \dim_k(L)$ and any Lie module N over L. In this case, the spectral sequence (5.1) is congruent, and $\mathrm{Ext}^n_{A^e\sharp\mathcal{U}(L)}(\Omega^1(A), M) = 0$ for large n. It follows from the long exact sequence in Theorem 5.3 that $\mathrm{H}^n_{LP}(A, L; M, P) = 0$ for sufficiently large n. □

Example 5.5. Let $A = \mathbb{M}_2(k)$ be the 2×2 full matrix algebra, $L = \mathfrak{sl}_2(k)$ be the symplectic algebra, and $\mu(x)(a) = [x, a] = xa - ax$ for $x \in L$, $a \in A$. Clearly, (A, L) is a Leibniz pair. We have the following simple facts:

$$\mathrm{Ext}^p_{A^e}(\Omega^1(A), A) = \mathrm{H}H^{p+1}(A) = 0 \qquad \text{for } p \geq 1,$$
$$\mathrm{H}om_{A^e}(\Omega^1(A), A) = \mathrm{Der}(A) \cong \mathfrak{sl}_2(k) = L.$$

By the spectral sequence (5.1), we have

$$\mathrm{Ext}^n_{A^e\sharp\mathcal{U}(L)}(\Omega^1(A), A) \cong \mathrm{Ext}^n_{\mathcal{U}(L)}(k, L) = \mathrm{H}L^n(L) = 0$$

for any $n \geq 0$, where the last equality follows from [6, Chapter VII, Proposition 6.1 and 6.3]. It follows from Theorem 5.3 that $\mathrm{H}^n_{LP}(A, L) = 0$ for any $n \geq 0$.

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